The natural logarithm is the logarithm to the base \( e \), where \( e \) is an irrational and transcendental constant approximately equal to 2.718281828. The natural logarithm is generally written as \( \ln(x) \), \( \log_e(x) \) or sometimes, if the base of \( e \) is implicit, as simply \( \log(x) \).[1]

The natural logarithm of a number \( x \) is the power to which \( e \) would have to be raised to equal \( x \). For example, \( \ln(7.389...) \) is 2, because \( e^2 = 7.389... \). The natural log of \( e \) itself (\( \ln(e) \)) is 1 because \( e^1 = e \), while the natural logarithm of 1 (\( \ln(1) \)) is 0, since \( e^0 = 1 \).

The natural logarithm can be defined for any positive real number \( a \) as the area under the curve \( y = 1/x \) from 1 to \( a \). The simplicity of this definition, which is matched in many other formulas involving the natural logarithm, leads to the term "natural." The definition can be extended to non-zero complex numbers, as explained below.

The natural logarithm function, if considered as a real-valued function of a real variable, is the inverse function of the exponential function, leading to the identities:

Like all logarithms, the natural logarithm maps multiplication into addition:

\[ e^{\ln(x)} = x \quad \text{if} \quad x > 0 \]

\[ \ln(e^x) = x. \]

Thus, the logarithm function is an isomorphism from the group of positive real numbers under multiplication to the group of real numbers under addition, represented as a function:

\[ \ln(xy) = \ln(x) + \ln(y) \]
Logarithms can be defined to any positive base other than 1, not just $e$; however logarithms in other bases differ only by a constant multiplier from the natural logarithm, and are usually defined in terms of the latter. Logarithms are useful for solving equations in which the unknown appears as the exponent of some other quantity. For example, logarithms are used to solve for the half-life, decay constant, or unknown time in exponential decay problems. They are important in many branches of mathematics and the sciences and are used in finance to solve problems involving compound interest.

**History**

The first mention of the natural logarithm was by Nicholas Mercator in his work *Logarithmotechnia* published in 1668,[2] although the mathematics teacher John Speidell had already in 1619 compiled a table on the natural logarithm.[3] It was formerly also called hyperbolic logarithm,[4] as it corresponds to the area under a hyperbola. It is also sometimes referred to as the Napierian logarithm, although the original meaning of this term is slightly different.

**Notational conventions**

The notations $\ln x$ and $\log_e x$ both refer unambiguously to the natural logarithm of $x$.

$\log x$ without an explicit base may also refer to the natural logarithm. This usage is common in some scientific contexts as well as in many programming languages.[5] $\log x$ is frequently used to denote the common (base 10) logarithm, however.

**Origin of the term natural logarithm**

Initially, it might seem that since the common numbering system is base 10, this base would be more "natural" than base $e$. But mathematically, the number 10 is not particularly significant. Its use culturally—as the basis for many societies' numbering systems—likely arises from humans' typical number of fingers.[6] Other cultures have based their counting systems on such choices as 5, 8, 12, 20, and 60.[7][8][9]

$\log_e$ is a "natural" log because it automatically springs from, and appears so often in, mathematics.

For example, consider the problem of differentiating a logarithmic function:[10]

If the base $b$ equals $e$, then the derivative is simply $1/x$, and at $x = 1$ this derivative equals 1.

Another sense in which the

$$
\frac{d}{dx} \log_b (x) = \frac{1}{\ln(b) \ln x} = \frac{1}{\ln(b)} \frac{d}{dx} \ln x = \frac{1}{x \ln(b)}
$$
base-e-logarithm is the most natural is that it can be defined quite easily in terms of a simple integral or **Taylor series** and this is not true of other logarithms.

Further senses of this naturalness make no use of **calculus**. As an example, there are a number of simple series involving the natural logarithm. **Pietro Mengoli** and **Nicholas Mercator** called it **logarithmus naturalis** a few decades before **Newton** and **Leibniz** developed calculus.[11]

## Definitions

Formally, \( \ln(a) \) may be defined as the **integral**,

\[
\ln(a) = \int_1^a \frac{1}{x} \, dx.
\]

This function is a logarithm because it satisfies the fundamental property of a logarithm:

This can be demonstrated by splitting the integral that defines \( \ln(ab) \) into two parts and then making the **variable substitution** \( x = ta \) in the second part, as follows:

\[
\ln(a) = \int_1^a \frac{1}{x} \, dx = \int_1^a \frac{1}{x} \, dx + \int_a^b \frac{1}{x} \, dx = \int_1^a \frac{1}{x} \, dx + \int_1^b \frac{1}{at} \, d(at)
\]

The number \( e \) can then be defined as the unique real number \( a \) such that \( \ln(a) = 1 \).

Alternatively, if the **exponential function** has been defined first, say by using an **infinite series**, the natural logarithm may be defined as its **inverse function**, i.e., \( \ln \) is that function such that \( \exp(\ln(x)) = x \). Since the range of the exponential function on real arguments is all positive real numbers and since the exponential function is strictly increasing, this is well-defined for all positive \( x \).

## Properties

- \( \ln(1) = 0 \)
• \( \ln(-1) = i\pi \) 

(see complex logarithm)

• \( \ln(x) < \ln(y) \) for \( 0 < x < y \)

• \( \frac{h}{1 + h} \leq \ln(1 + h) \leq h \) for \( h > -1 \)

• \( \lim_{x \to 0} \frac{\ln(1 + x)}{x} = 1. \)

**Derivative, Taylor series**

The derivative of the natural logarithm is given by

\[
\frac{d}{dx} \ln(x) = \frac{1}{x}. 
\]

This leads to the Taylor series for \( \ln(1 + x) \) around 0; also known as the Mercator series

\[
\ln(1 + x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \quad \text{for} \quad |x| \leq 1
\]

At right is a picture of \( \ln(1 + x) \) and some of its Taylor polynomials around 0. 

unless \( x = -1 \)
These approximations converge to the function only in the region \(-1 < x \leq 1\); outside of this region the higher-degree Taylor polynomials are worse approximations for the function.

Substituting \(x - 1\) for \(x\), we obtain an alternative form for \(\ln(x)\) itself, namely

\[
\ln(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (x - 1)^n
\]

By using the Euler transform on the Mercator series, one obtains the following, which is valid for any \(x\) with absolute value greater than 1:

\[
\ln(x) = (x - 1) - \frac{(x - 1)^2}{2} + \frac{(x - 1)^3}{3} - \frac{(x - 1)^4}{4} \ldots
\]

This series is similar to a BBP-type formula.

Also note that is its own inverse function, so to yield the natural logarithm of a certain number \(y\), simply put in for \(x\).

\[
\ln \frac{x}{x - 1} = \sum_{n=1}^{\infty} \frac{1}{nx^n} = \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{3x^3} + \ldots
\]

### The natural logarithm in integration

The natural logarithm allows simple integration of functions of the form \(g(x) = f'(x)/f(x)\): an antiderivative of \(g(x)\) is given by \(\ln(|f(x)|)\). This is the case because of the chain rule and the following fact:

\[
\frac{d}{dx} (\ln |x|) = \frac{1}{x}.
\]

In other words,

\[
\frac{d}{dx} (\ln |x|) = \frac{1}{x}.
\]

Here is an example in the case of \(g(x) = \tan(x)\):

Letting \(f(x) = \cos(x)\) and \(f'(x) = -\sin(x)\):

\[
\int \frac{1}{x} dx = \ln |x| + C
\]

where \(C\) is an arbitrary constant of integration.

The natural logarithm can be integrated using integration by parts:

\[
\int \tan(x) \, dx = \int \frac{\sin(x)}{\cos(x)} \, dx
\]

### Numerical value

To calculate the numerical value of the natural logarithm of a number, the Taylor series expansion can be rewritten as:

\[
\int \tan(x) \, dx = \int -\frac{d}{dx} \frac{\cos(x)}{\cos(x)} \, dx
\]
To obtain a better rate of convergence, the following identity can be used.

\[ \ln(1+x) = x \left( \frac{1}{1} - x \left( \frac{1}{2} - x \left( \frac{1}{3} - x \left( \frac{1}{4} - x \left( \frac{1}{5} - \cdots \right) \right) \right) \right) \right) \quad \text{for} \quad |x| < 1. \]

To obtain a better rate of convergence, the following identity can be used.

\[ \ln(x) = \ln \left( \frac{1+y}{1-y} \right) = 2y \left( \frac{1}{1} + \frac{1}{3}y^2 + \frac{1}{5}y^4 + \frac{1}{7}y^6 + \frac{1}{9}y^8 + \cdots \right) \]

provided that \( y = (x-1)/(x+1) \) and \( x > 0 \).

For \( \ln(x) \) where \( x > 1 \), the closer the value of \( x \) is to 1, the faster the rate of convergence. The identities associated with the logarithm can be leveraged to exploit this:

\[ \ln(123.456) = \ln(1.23456 \times 10^2) \]

\[ = \ln(1.23456) + \ln(10^2) \]

\[ = \ln(1.23456) + 2 \times \ln(10) \]

\[ \approx \ln(1.23456) + 2 \times 2.3025851 \]
Such techniques were used before calculators, by referring to numerical tables and performing manipulations such as those above.

**Natural logarithm of 10**

The natural logarithm of 10 (A002392) plays a role for example in computation of natural logarithms of numbers represented in the scientific notation, a mantissa multiplied by a power of 10:

\[
\ln(a \times 10^n) = \ln a + n \ln 10.
\]

By this scaling, the algorithm may reduce the logarithm of all positive real numbers to an algorithm for natural logarithms in the range \(1 \leq a < 10\).

**High precision**

To compute the natural logarithm with many digits of precision, the Taylor series approach is not efficient since the convergence is slow. An alternative is to use Newton's method to invert the exponential function, whose series converges more quickly.

An alternative for extremely high precision calculation is the formula [13] [14]

\[
\ln x \approx \frac{\pi}{2M(1, 4/s)} - m \ln 2
\]

with \(m\) chosen so that \(p\) bits of precision is attained. (For most purposes, the value of 8 for \(m\) is sufficient.) In fact, if this method is used, Newton inversion of the natural logarithm may conversely be used to calculate the exponential function efficiently. (The constants \(\ln 2\) and \(\pi\) can be pre-computed to the desired precision using any of several known quickly converging series.)

**Computational complexity**

The computational complexity of computing the natural logarithm (using the arithmetic-geometric mean) is \(O(M(n) \ln n)\). Here \(n\) is the number of digits of precision at which the natural logarithm is to be evaluated and \(M(n)\) is the computational complexity of multiplying two \(n\)-digit numbers.

**Continued fractions**

While no simple continued fractions are available, several generalized continued fractions are, including:
Complex logarithms

Main article: Complex logarithm

The exponential function can be extended to a function which gives a complex number as $e^x$ for any arbitrary complex number $x$; simply use the infinite series with $x$ complex. This exponential function can be inverted to form a complex logarithm that exhibits most of the properties of the ordinary logarithm. There are two difficulties involved: no $x$ has $e^x = 0$; and it turns out that $e^{2\pi i} = 1 = e^0$. Since the multiplicative property still works for the complex exponential function, $e^z = e^{z + 2\pi in}$, for all complex $z$ and integers $n$.

So the logarithm cannot be defined for the whole complex plane, and even then it is multi-valued – any complex logarithm can be changed into an "equivalent" logarithm by adding any integer multiple of $2\pi i$ at will. The complex logarithm can only be single-valued on the cut plane. For example, $\ln i = 1/2 \pi i$ or $5/2 \pi i$ or $-3/2 \pi i$, etc.; and although $i^4 = 1$, $4 \log i$ can be defined as $2\pi i$, or $10\pi i$ or $-6 \pi i$, and so on.

- Plots of the natural logarithm function on the complex plane (principal branch)

\[
\ln(1 + x) = \frac{x^1}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots = \frac{x}{1 - 0x + \frac{1^2x}{2 - 1x + \frac{2^2x}{3 - 2x + \frac{3^2x}{4 - 3x + \frac{4^2x}{5 - 4x + \cdots}}}}
\]

\[
\ln \left(1 + \frac{2x}{y}\right) = \frac{2x}{y + \frac{x}{1 + \frac{2x}{3y + \frac{2x}{1 + \frac{3x}{5y + \frac{3x}{1 + \cdots}}}}}} = \frac{2x}{y + x - \frac{(1x)^2}{3(y + x) - \frac{(2x)^2}{5(y + x) - \frac{(3x)^2}{7(y + x) - \cdots}}}}
\]
\[ z = |\text{Im}(\ln(x+iy))| \]

\[ z = |\ln(x+iy)| \]

Superposition of the previous 3 graphs

See also

- John Napier – inventor of logarithms
- Logarithm of a matrix
- Logarithmic integral function
- Nicholas Mercator – first to use the term natural log
- Polylogarithm
- Von Mangoldt function
- The number \( e \)
- Natural logarithm of 2
- Leonhard Euler

References


2. ^ J J O’Connor and E F Robertson (2001-09). "The number \( e \)." The MacTutor History of


5. ^ Including C, C++, SAS, MATLAB, Mathematica, Fortran, and BASIC


12. ^ "Logarithmic Expansions" at Math2.org


External links

- Demystifying the Natural Logarithm (In) | BetterExplained